Itô's Lemma
3 Questions & Answers

1st and 2nd Questions: Application of Itô's Lemma
3rd Question: Itô's Lemma and Sharpe Ratio

Q1. $\int_0^t W_s \, dW_s$

Solution:

Since W_t is Standard Brownian motion then $Y_t = W_t{}^2$ satisfies Itô's Lemma. So;

$$dY_t = 2W_t dW_t + \frac{1}{2} 2dt$$

$$\frac{1}{2}(dY_t - dt) = W_t dW_t.$$

By taking the integration for both sides, the result follows:

$$\frac{1}{2}\int_0^t (dY_s - ds) = \frac{1}{2}(Y_t - t) = \frac{1}{2}\left(W_t{}^2 - t\right) = \int_0^t W_s \, dW_s$$

Q2. Let B_t be a standard Brownian motion in a the Black-Scholes framework and $Y_t = t^2 B_t^4$ be a solution of a stochastic differential equation. Find the SDE.

Solution:

Since Y_t satisfies Itô's Lemma, then:

$$dY_t = d(t^2 B_t^4) = t^2 4 B_t^3 dB_t + \frac{1}{2} t^2 12 B_t^2 dB_t^2 + 2t B_t^4 dt$$

$$dY_t = (6t^2 B_t^2 + 2t B_t^4) dt + 4t^2 B_t^3 dB_t$$

$$dY_t = \left(6t Y_t^{1/2} + 2\frac{Y_t}{t}\right) dt + 4t^{1/2} Y_t^{3/4} dB_t$$

Arranging the terms concludes the result.

$$dY_t = \left(6(t^2 Y_t)^{1/2} + 2\frac{Y_t}{t}\right) dt + 4\left(t^{2/3} Y_t\right)^{3/4} dB_t$$

Q3. Two non-dividend paying stocks, X ve Y have the same source of uncertainty as W_t, a standard Brownian motion izlemektedir. Price of the assets follows the below stochastic differential equations:

$$\frac{dX_t}{X_t} = 0.07dt + 0.12dZ_t$$

$$\frac{dY_t}{Y_t} = Adt + BdZ_t$$

The equality $d[lnY_t] = \alpha dt + 0.085dZ_t$ is given and the continuous interest rate is assumed as 4%. Find the constant A.

Solution:

Since Y_t is a geometric Brownian motion, the function $G_t = lnY_t$ satisfies Itô's Lemma. Then

$$d[lnY_t] = \frac{1}{Y_t}dY_t + \frac{1}{2}\left(-\frac{1}{Y_t^2}\right)(dY_t)^2$$

$$d[lnY_t] = Adt + BdZ_t - \frac{1}{2}B^2dt$$

$$d[lnY_t] = \left(A - \frac{1}{2}B^2\right)dt + BdZ_t$$

and $B = 0.085$.

Since they have the same source of uncertainty, they have the same Sharpe ratios.

So;

$$\frac{\mu_X - r}{\sigma_X} = \frac{\mu_Y - r}{\sigma_Y}$$

$$\frac{0.07 - 0.04}{0.12} = \frac{A - 0.04}{0.085}$$

$$A = 0.06125.$$